Coefficient Estimates and Fekete-Szegö Inequalities for New Subclasses of Bi-Univalent Functions Defined by Convolution Operator

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Abstract: This study presents a new convolution operator constructed through the Hadamard product of the Komatu integral operator with the Opoola differential operator. By applying quasi-subordination, two new subclasses of bi-univalent functions are defined in the open unit disk U. For these subclasses, estimates are obtained for the coefficients $|\mathcal{d}_2|$ and $|\mathcal{d}_3|$. In addition, the Fekete-Szegö inequalities are established for the introduced subclasses. Several known results are recovered as special cases by suitable choices of the involved parameters.

Keywords: Bi-Univalent Functions, Quasi-Subordination, Convolution Operator, Coefficient Estimates, Fekete-Szegö.

1-Introduction

Let $\mathcal{A}$ reprsent the class of all normalized analytic functions in the open unit disk $U = \{\xi: |\xi| < 1\}$, with Taylor expansion

$$\mathcal{T}(\xi) = \xi + \sum_{n=2}^{\infty} \mathcal{d}_n \xi^n , \qquad (\xi \in U). \tag{1.1}$$

Let $\boldsymbol{\mathcal{S}}$ represen the subclass of all functions in $\mathcal{A}$ which are univalent in U.

The class of univalent functions has been extensively investigated because of its fundamental role in geometric function theory.

By the Koebe one-quarter theorem [1], the inverse $(\mathcal{T}^{-1})$ of every $\mathcal{T} \in \mathcal{S}$ satisfies the following relations:

$$\mathcal{T}^{-1}\big(\mathcal{T}(\xi)\big) = \xi. \qquad (\xi \in U)$$

and

$$\mathcal{T}\left(\mathcal{T}^{-1}(\zeta)\right) = \zeta \qquad \left(|\zeta| < r_0(\mathcal{T}), r_0(\mathcal{T}) \geq \frac{1}{4}\right),$$

The inverse function admits the next Taylor-Maclaurin expansion:

$$\begin{aligned}\mathcal{T}^{-1}(\zeta) = \mathcal{R}(\zeta) \\ &= \zeta - d_2\zeta^2 + (2d_2^2 - d_3)\zeta^3 - (5d_2^3 - 5d_2d_3 + d_4)\zeta^4 \\ &+ \cdots, \quad (1.2)\end{aligned}$$

A function $\mathcal{T}$ is bi-univalent in U when both $\mathcal{T}$ and $\mathcal{T}^{-1}$ are univalent in there.

We use $\sum$ to denote The class of all bi-univalent functions in U.

Consider $\mathcal{T}, \mathcal{R} \in \mathcal{A}$, where $\mathcal{T}$ be given by (1.1), and

$$\mathcal{R}(\xi) = \xi + \sum_{n=2}^{\infty} b_n \xi^n, \qquad (\xi \in U). \qquad (1.3)$$

Then the convolution product of $\mathcal{T}$ and $\mathcal{R}$, by $\mathcal{T}(\xi) * \mathcal{R}(\xi)$, is given by

$$\mathcal{T}(\xi) * \mathcal{R}(\xi) = \xi + \sum_{n=2}^{\infty} d_n b_n \xi^n, \qquad (\xi \in U). \qquad (1.4)$$

The Hadamard product also known as convolution is an important tool in geometric function theory and has been used to construct new operators and investigate various subclasses of analytic and bi-univalent functions.

Among these operators, the Komatu integral operator has been widely used.

The Komatu integral operator, introduced by Komatu [2], is defined by:

$$K_{\varsigma}^{\varrho}\mathcal{T}(\xi) = \xi + \sum_{n=2}^{\infty} \left(\frac{\varsigma + 1}{\varsigma + n}\right)^{\varrho} d_n \xi^n, \quad (\varsigma > 0, \varrho \geq 0, \xi \in U)$$

Another important operator is the Opoola differential operator, introduced by Opoola [3], which is defined as follows:

$$D^{\eta}(\mu,\rho,t)\mathcal{T}(\xi) = \xi + \sum_{n=2}^{\infty}(1+(n+\rho-\mu-1)t)^{\eta} d_n \xi^n ,$$

$$(\eta \in N_0, \rho \geq \mu, \mu > 0, t \geq 0, \xi \in U)$$

Using the Hadamard product of the Komatu integral operator $K_{\varsigma}^{\varrho}\mathcal{T}(\xi)$ with the Opoola differential operator $D^{\eta}(\mu,\rho,t)\mathcal{T}(\xi)$, we define the convolution operator:

$$\begin{aligned}\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi) &= K_{\varsigma}^{\varrho}\mathcal{T}(\xi) * D^{\eta}(\mu,\rho,t)\mathcal{T}(\xi) \\ &= \xi \\ &+ \sum_{n=2}^{\infty}\left(\frac{\varsigma+1}{\varsigma+n}\right)^{\varrho}(1+(n+\rho-\mu-1)t)^{\eta} d_n \xi^n , \qquad (1.5)\end{aligned}$$

For simplicity, the above operator can be rewritten as:

$$\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi) = \xi + \sum_{n=2}^{\infty}\mathcal{Q}_{\varsigma,n}^{\mu} d_n \xi^n ,$$

where $\mathcal{Q}_{\varsigma,n}^{\mu} = \left(\frac{\varsigma+1}{\varsigma+n}\right)^{\varrho}(1+(n+\rho-\mu-1)t)^{\eta}$

Remark$(\mathbf{1.1})$: By assigning suitable values to the parameters, the operator defined in (1.5) akes the form of the following established operators:

i. When $\eta = 0$, the new convolution operator reduces to the Komatu integral operator, that is

$$\mathcal{H}_{\mu,\varsigma,t}^{\varrho,0,\rho}\mathcal{T}(\xi) = \xi + \sum_{n=2}^{\infty}\left(\frac{\varsigma+1}{\varsigma+n}\right)^{\varrho} d_n \xi^n = K_{\varsigma}^{\varrho}\mathcal{T}(\xi)$$

ii. When $\varrho = 0$, the new convolution operator reduces to the Opoola differential operator, that is

$$\mathcal{H}_{\mu,\varsigma,t}^{0,\eta,\rho}\mathcal{T}(\xi) = \xi + \sum_{n=2}^{\infty}(1+(n+\rho-\mu-1)t)^{\eta} d_n \xi^n = D^{\eta}(\mu,\rho,t)\mathcal{T}(\xi)$$

Hence, the proposed operator includes the Komatu integral operator and the Opoola differential operator as special cases.

Motivated by these considerations, we employ the proposed convolution operator together with quasi-subordination to introduce new subclasses of bi-univalent functions and investigate their initial Taylor–Maclaurin coefficient estimates and Fekete–Szegö inequalities.

If $\mathcal{T}$ and $\mathcal{R}$ be analytic in U. We say that $\mathcal{T}$ is quasi-subordinate to $\mathcal{R}$ in U, and write:

$$\mathcal{T} \prec_q \mathcal{R} \ \ or \ \ \mathcal{T}(\xi) \prec_q \mathcal{R}(\xi), \qquad (\xi \in U).$$

Suppose that analytic functions Ψ and ζ exist, satisfying $|\Psi(\xi)| < 1,\ \zeta(0) = 0$ and $|\zeta(\xi)| < 1$ such that $\mathcal{T}(\xi) = \Psi(\xi)\mathcal{R}\big(\zeta(\xi)\big)$, for all $(\xi \in U)$.

In fact, if $\Psi(\xi) = 1$, then $\mathcal{T}(\xi) = \mathcal{R}\big(\zeta(\xi)\big)$ so that $\mathcal{T}(\xi) \prec \mathcal{R}(\xi)$ in U. Also, if $\zeta(\xi) = \xi$, then $\mathcal{T}(\xi) = \Psi(\xi)\mathcal{R}(\xi)$, and it is said that $\mathcal{T}$ is majorized to ϑ, written $\mathcal{T}(\xi) \ll \mathcal{R}(\xi)$ in U. for more information see ([4], [10]).

Lewin [11] introduced the concept of the class $\sum$ of bi-univalent functions in 1967 and showed that $|d_2| \leq 1.51$. Moreover, Brannan and Clunie [12] conjectured that $|d_2| \leq \sqrt{2}$, and Netanyahu [13] obtained that $md\xi_{\mathcal{T}\in\sum}|d_2| = \frac{4}{3}$ Later,

Styer and Wright [14] showed that there exists a function $\mathcal{T}(\xi)$ so that $|d_2| > \frac{4}{3}$. However, Tan [15] obtained the upper bound estimate $|d_2| < 1.485$ of coefficient for function $\sum$ by based on the works of Brannan and Taha [16] and Srivastava et al. [17], numerous researchers examined classes of bi-univalent analytic functions and discovered estimates for the Taylor-Maclaurin coefficients $|d_2|$ and $|d_3|$ for each function in these classes. [18], [19].

Ma and Minda [20] introduced and studied classes $S^*(\Psi)$ and $\mathcal{R}^*(\Psi)$ by means of subordination, defined as follows:

$$S^*(\Psi) = \left\{\mathcal{T} \in \mathcal{A}: \frac{\xi\,\mathcal{T}'(\xi)}{\mathcal{T}(\xi)} \prec \Psi(\xi), \qquad \xi \in U\right\},$$

and

$$G^*(\Psi) = \left\{\mathcal{T} \in \mathcal{A}: 1 + \frac{\xi\,\mathcal{T}''(\xi)}{\mathcal{T}(\xi)} \prec \Psi(\vartheta), \qquad \xi \in U\right\},$$

The classes $S^*(\Psi)$ and $G^*(\Psi)$ are called bi-Starlike and bi-convex of Ma-Minda type, respectively.

In geometric function theory, the Fekete-Szegö functional $|d_3 - \mu d_2^2|$ for normalized univalent functions of the form defined by (1.1) has a long and well-established history. The Fekete-Szegö functional has garnered a lot of interest in recent years, especially in relation to numerous subclasses of the class of normalized analytic and univalent functions [21].

Now, consider

$$\vartheta(\xi) = 1 + C_1\xi + C_2\xi^2 + C_3\xi^3 + \cdots, \qquad (C_1 > 0) \tag{1.6}$$

where $\vartheta(0) = 1$ and $\vartheta'(0) > 0$ also

$$\Psi(\xi) = k_0 + k_1\xi + k_2\xi^2 + \cdots, \tag{1.7}$$

The following lemma will be required in establishing our main results

Lemma $(\mathbf{1.1})$ [1]: If $p \in P$, then $|p_i| \leq 2$ for each i, where p is the family of all functions p, analytic in U, for which

$$Re\{p(\xi)\} > 0, \qquad (\xi \in U),$$

where

$$p(\xi) = 1 + p_1\xi + p_2\xi^2 + p_3\xi^3 + \cdots, \qquad (\xi \in U).$$

Lemma$(\mathbf{1.2})$[20]: Let ζ be analytic function in U, with $\zeta(0) = 0, |\zeta(\xi)| < 1$ and

$$\zeta(\xi) = \zeta_1\xi + \zeta_2\xi^2 + \zeta_3\xi^3 + \cdots.$$

then

$$\left|\zeta_2 + \mu\zeta_1^2\right| \leq \max\{1, |\mu|\}, \mu \in \mathbb{C}.$$

The result is sharp for the function $\zeta(\xi) = \xi^2$ or $\zeta(\xi) = \xi$.

2- The Subclass $\mathcal{N}_{q,\Sigma}^{\varepsilon,\alpha}(\hbar, \delta, \vartheta)$

Definition $(\mathbf{2.1})$: Let $\mathcal{T} \in \mathcal{A}$ be in the form (1.1) is said to belong to the class $\mathcal{N}_{q,\Sigma}^{\varepsilon,\alpha}(\hbar, \delta, \vartheta)$, provide that the following quasi-subordination conditions hold:

$$\frac{1}{\hbar}\left[\left(\frac{\xi\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)'}{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)}\right)\left(\frac{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)}{\xi}\right)^{\varepsilon} + \left\{\frac{\xi\left(\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)'\right)^{\alpha}}{\delta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)+(1-\delta)\xi}-1\right\}\right] \prec_q (\vartheta(\xi)-1), \qquad (2.1)$$

and

$$\frac{1}{\hbar}\left[\left(\frac{\zeta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)'}{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)}\right)\left(\frac{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)}{\zeta}\right)^{\varepsilon} + \left\{\frac{\zeta\left(\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)'\right)^{\alpha}}{\delta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)+(1-\delta)\zeta}-1\right\}\right] \prec_q (\vartheta(\zeta)-1), \qquad (2.2)$$

where $\varepsilon \geq 0, 0 \leq \hbar \leq 1, \alpha \geq 1, \delta \in \mathbb{C}, \mathcal{R} = \mathcal{T}^{-1}$, ϑ is given by (1.3), and $\xi, \zeta \in U$.

Taking $\varepsilon = 0$ in Definition (2.1), yields the following special case.

Remark(2.1): A function $\mathcal{T} \in \mathcal{A}$ be defined by (1.1). Then $\mathcal{T}$ be in the class $\mathcal{N}_{q,\Sigma}^{\alpha}(\hbar, \delta, \vartheta)$, whenever the following quasi-subordination conditions hold:

$$\frac{1}{\hbar}\left[\left(\frac{\xi\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)'}{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)}\right) + \left\{\frac{\xi\left(\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)'\right)^{\alpha}}{\delta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)+(1-\delta)\xi}-1\right\}\right] \prec_q (\vartheta(\xi)-1).$$

and

$$\frac{1}{\hbar}\left[\left(\frac{\zeta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)'}{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)}\right) + \left\{\frac{\zeta\left(\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)'\right)^{\alpha}}{\delta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)+(1-\delta)\zeta}-1\right\}\right] \prec_q (\vartheta(\zeta)-1).$$

Theorem (2.1): For $\mathcal{T}$ defined by (1.1) and belonging to the class $\mathcal{N}_{q,\Sigma}^{\varepsilon,\alpha}(\hbar, \delta, \vartheta)$, then

$$|d_2| \leq \frac{\hbar|\mathcal{P}_0|\mathcal{C}_1\sqrt{\mathcal{C}_1}}{\sqrt{\begin{matrix}\hbar\mathcal{P}_0\mathcal{C}_1^2(2+\varepsilon+3\alpha-\delta)\mathcal{Q}_{\varsigma,3}^{\mu}+\\ \left[\hbar\mathcal{P}_0\mathcal{C}_1^2\left(\frac{1}{2}(\varepsilon^2+\varepsilon-2)-\delta(2\alpha-\delta)+2\alpha(\alpha-1)\right)\right]\mathcal{Q}_{\varsigma,2}^{\mu,2}\\ -\left((\mathcal{C}_2-\mathcal{C}_1)(1+\varepsilon+2\alpha-\delta)^2\right)\end{matrix}}}, \quad (2.3)$$

and

$$|d_3| \leq \frac{\hbar\mathcal{C}_1(|\mathcal{P}_0|+|\mathcal{P}_1|)}{(2+\varepsilon+3\alpha-\delta)\mathcal{Q}_{\varsigma,3}^{\mu}} + \frac{\hbar^2\mathcal{P}_0^2\mathcal{C}_1^2}{(1+\varepsilon+2\alpha-\delta)^2\mathcal{Q}_{\varsigma,2}^{\mu,2}}, \quad (2.4)$$

where

$\mathcal{Q}_{\varsigma,2}^{\mu} = \left(\frac{\varsigma+1}{\varsigma+2}\right)^{\varrho}(1+(2+\rho-\mu-1)t)^{\eta}$ and

$\mathcal{Q}_{\varsigma,3}^{\mu} = \left(\frac{\varsigma+1}{\varsigma+2}\right)^{\varrho}(1+(3+\rho-\mu-1)t)^{\eta}$

Proof: Since $\mathcal{T} \in \mathcal{N}_{q,\Sigma}^{\varepsilon,\alpha}(\hbar,\delta,\vartheta)$, there are analytic functions $\upsilon, \nu: U \to U$ with $\upsilon(0) = \nu(0) = 0$, $|\upsilon(\xi)| < 1$ and $|\nu(\zeta)| < 1$ such that:

$$\frac{1}{\hbar}\left[\left(\frac{\xi\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)'}{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)}\right)\left(\frac{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)}{\xi}\right)^{\varepsilon} + \left\{\frac{\xi\left(\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)'\right)^{\alpha}}{\delta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)+(1-\delta)\xi} - 1\right\}\right] = \Psi(\xi)\left(\vartheta(\upsilon(\xi)-1)\right), \quad (2.5)$$

and

$$\frac{1}{\hbar}\left[\left(\frac{\zeta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)'}{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)}\right)\left(\frac{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)}{\zeta}\right)^{\varepsilon} + \left\{\frac{\zeta\left(\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)'\right)^{\alpha}}{\delta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)+(1-\delta)\zeta} - 1\right\}\right] = \Psi(\zeta)\left(\vartheta(\nu(\zeta)-1)\right) \quad (2.6)$$

where $\mathcal{R} = \mathcal{T}^{-1}$ and $\xi, \zeta \in U$.

Let $\wp(\xi)$ and $q(\zeta)$ be given by

$$\wp(\xi) = \frac{1+\upsilon(\xi)}{1-\upsilon(\xi)} = 1 + \mathcal{s}_1\xi + \mathcal{s}_2\xi^2 + \cdots, \tag{2.7}$$

and

$$q(\zeta) = \frac{1+\nu(\zeta)}{1-\nu(\zeta)} = 1 + \mathcal{t}_1\zeta + \mathcal{t}_2\zeta^2 + \cdots, \tag{2.8}$$

or equivalently

$$\upsilon(\xi) = \frac{\wp(\xi)-1}{\wp(\xi)+1} = \frac{1}{2}\left[\mathcal{s}_1\xi + \left(\mathcal{s}_2 - \frac{\mathcal{s}_1^2}{2}\right)\xi^2 + \cdots\right]. \tag{2.9}$$

and

$$\nu(\zeta) = \frac{q(\zeta)-1}{q(\zeta)+1} = \frac{1}{2}\left[\mathcal{t}_1\zeta + \left(\mathcal{t}_2 - \frac{\mathcal{t}_1^2}{2}\right)\zeta^2 + \cdots\right]. \tag{2.10}$$

The functions $\wp(\xi)$ and $q(\zeta)$ are analytic in U and satisfy $\wp(0) = q(0) = 1$. Moreover, since $\upsilon, \nu: U \to U$, both $\wp(\xi)$ and $q(\zeta)$ have positive real parts throughout U.

Using (2.9) and (2.10) in (2.5) and (2.6) respectively, we get:

$$\frac{1}{\hbar}\left[\left(\frac{\xi\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)'}{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)}\right)\left(\frac{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)}{\xi}\right)^{\varepsilon} + \left\{\frac{\xi\left(\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)'\right)^{\alpha}}{\delta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right) + (1-\delta)\xi} - 1\right\}\right]$$
$$= \Psi(\xi)\left(\vartheta\left(\frac{\wp(\xi)-1}{\wp(\xi)+1}\right) - 1\right). \tag{2.11}$$

and

$$\frac{1}{\hbar}\left[\left(\frac{\zeta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)'}{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)}\right)\left(\frac{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)}{\zeta}\right)^{\varepsilon} + \left\{\frac{\zeta\left(\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)'\right)^{\alpha}}{\delta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right) + (1-\delta)\zeta} - 1\right\}\right]$$
$$= \Psi(\zeta)\left(\vartheta\left(\frac{q(\zeta)-1}{q(\zeta)+1}\right) - 1\right). \tag{2.12}$$

Using the series expansions for $\mathcal{T}(\xi)$ and $\mathcal{R}(\zeta)$ given by (1.1) and (1.3) respectively, we obtain

$$\frac{1}{\hbar}\left[\left(\frac{\xi\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)'}{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)}\right)\left(\frac{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)}{\xi}\right)^{\varepsilon}+\left\{\frac{\xi\left(\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)'\right)^{\alpha}}{\delta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)+(1-\delta)\xi}-1\right\}\right]$$
$$=\frac{1}{\hbar}\Bigg[1+(1+\varepsilon+2\varepsilon-\delta)\mathcal{Q}_{\varsigma,2}^{\mu}d_2\xi$$
$$+\Bigg((2+\varepsilon+3\alpha-\delta)\mathcal{Q}_{\varsigma,3}^{\mu}d_3$$
$$+\left(\frac{1}{2}(\varepsilon^2+\varepsilon-2)-\delta(2\alpha-\delta)+2\alpha(\alpha-1)\right)\mathcal{Q}_{\varsigma,2}^{\mu,2}d_2^2\Bigg)\xi^2$$
$$+\cdots\Bigg] \qquad (2.13)$$

and

$$\frac{1}{\hbar}\left[\left(\frac{\zeta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)'}{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)}\right)\left(\frac{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)}{\zeta}\right)^{\varepsilon}+\left\{\frac{\zeta\left(\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)'\right)^{\alpha}}{\delta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)+(1-\delta)\zeta}-1\right\}\right]$$
$$=\frac{1}{\hbar}\Bigg[1-(1+\varepsilon+2\alpha-\delta)\mathcal{Q}_{\varsigma,2}^{\mu}d_2\zeta$$
$$+\Bigg((2+\varepsilon+3\alpha-\delta)\mathcal{Q}_{\varsigma,3}^{\mu}\left(2d_2^2-d_3\right)$$
$$+\left(\frac{1}{2}(\varepsilon^2+\varepsilon-2)-\delta(2\alpha-\delta)+2\alpha\ \ (\alpha-1)\right)\mathcal{Q}_{\varsigma,2}^{\mu,2}d_2^2\Bigg)\zeta^2$$
$$+\cdots\Bigg] \qquad (2.14)$$

Using (2.9) and (2.10) together with (1.6) and (1.7), it is evident that

$$\Psi(\xi)\left(\vartheta\left(\frac{\mathcal{p}(\xi)-1}{\mathcal{p}(\xi)+1}\right)-1\right)$$
$$=\frac{1}{2}\mathcal{P}_0\mathcal{C}_1\mathcal{s}_1\xi+\left(\frac{1}{2}\mathcal{P}_1\mathcal{C}_1\mathcal{s}_1+\frac{1}{2}\mathcal{P}_0\mathcal{C}_1\left(\mathcal{s}_2-\frac{\mathcal{s}_1^2}{2}\right)+\frac{1}{4}\mathcal{P}_0\mathcal{C}_2\mathcal{s}_1^2\right)\xi^2$$
$$+\cdots, \qquad (2.15)$$

and

$$\Psi(\zeta)\left(\vartheta\left(\frac{q(\zeta)-1}{q(\zeta)+1}\right)-1\right) = \frac{1}{2}\mathcal{P}_0\mathcal{C}_1 t_1\zeta + \left(\frac{1}{2}\mathcal{P}_1\mathcal{C}_1 t_1 + \frac{1}{2}\mathcal{P}_0\mathcal{C}_1\left(t_2 - \frac{t_1^2}{2}\right) + \frac{1}{4}\mathcal{P}_0\mathcal{C}_2 t_1^2\right)\zeta^2 + \cdots, \quad (2.16)$$

Equating the coefficients of ξ and ξ^2 in (2.13) and (2.15), we obtain

$$\frac{1}{\hbar}(1+\varepsilon+2\alpha-\delta)\mathcal{Q}^{\mu}_{\varsigma,2}d_2 = \frac{1}{2}\mathcal{P}_0\mathcal{C}_1 s_1 \quad (2.17)$$

and

$$\frac{1}{\hbar}\Big[(2+\varepsilon+3\alpha-\delta)\mathcal{Q}^{\mu}_{\varsigma,3}d_3 + \left(\frac{1}{2}(\varepsilon^2+\varepsilon-2)-\delta(2\alpha-\delta)+2\alpha(\alpha-1)\right)\mathcal{Q}^{\mu,2}_{\varsigma,2}d_2^2\Big] = \frac{1}{2}\mathcal{P}_1\mathcal{C}_1 s_1 + \frac{1}{2}\mathcal{P}_0\mathcal{C}_1\left(s_2 - \frac{s_1^2}{2}\right) + \frac{1}{4}\mathcal{P}_0\mathcal{C}_2 s_1^2. \quad (2.18)$$

Similarly, the coefficients of ζ and ζ^2 in (2.14) and (2.16), yield

$$-\frac{1}{\hbar}(1+\varepsilon+2\alpha-\delta)\mathcal{Q}^{\mu}_{\varsigma,2}d_2 = \frac{1}{2}\mathcal{P}_0\mathcal{C}_1 t_1 \quad (2.19)$$

and

$$\frac{1}{\hbar}\Big((2+\varepsilon+3\alpha-\delta)\mathcal{Q}^{\mu}_{\varsigma,3}(2d_2^2-d_3) + \left(\frac{1}{2}(\varepsilon^2+\varepsilon-2)-\delta(2\alpha-\delta)+2\alpha\ (\alpha-1)\right)\mathcal{Q}^{\mu,2}_{\varsigma,2}d_2^2\Big) = \frac{1}{2}\mathcal{P}_1\mathcal{C}_1 t_1 + \frac{1}{2}\mathcal{P}_0\mathcal{C}_1\left(t_2 - \frac{t_1^2}{2}\right) + \frac{1}{4}\mathcal{P}_0\mathcal{C}_2 t_1^2 \quad (2.20)$$

From (2.17) and (2.19), we find

$$d_2 = \frac{\hbar \mathcal{P}_0 \mathcal{C}_1 s_1}{2(1+\varepsilon+2\alpha-\delta)\mathcal{Q}^{\mu}_{\varsigma,2}} = -\frac{\hbar \mathcal{P}_0 \mathcal{C}_1 t_1}{2(1+\varepsilon+2\alpha-\delta)\mathcal{Q}^{\mu}_{\varsigma,2}} \tag{2.21}$$

It follows that

$$s_1 = -t_1 \tag{2.22}$$

and

$$8(1+\varepsilon+2\alpha-\delta)^2 \mathcal{Q}^{\mu,2}_{\varsigma,2} d_2^2 = \hbar^2 \mathcal{P}_0^2 \mathcal{C}_1^2 \left(s_1^2 + t_1^2\right) \tag{2.23}$$

Applying Lemma (1.1) in (2.21) we obtain

$$|d_2| \leq \frac{\hbar |\mathcal{P}_0| \mathcal{C}_1}{(1+\varepsilon+2\alpha-\delta)\mathcal{Q}^{\mu}_{\varsigma,2}}.$$

Adding (2.18) and (2.20), and using (2.22) and (2.23), we obtain

$$\begin{aligned} 4\Bigg[\hbar \mathcal{P}_0 \mathcal{C}_1^2 \Bigg(& (2+\varepsilon+3\alpha-\delta)\mathcal{Q}^{\mu}_{\varsigma,3} \\ & + \left(\frac{1}{2}(\varepsilon^2+\varepsilon-2) - \delta(2\alpha-\delta) + 2\alpha(\alpha-1)\right)\mathcal{Q}^{\mu,2}_{\varsigma,2}\Bigg) \\ & - \left((\mathcal{C}_2-\mathcal{C}_1)(1+\varepsilon+2\alpha-\delta)^2\right)\mathcal{Q}^{\mu,2}_{\varsigma,2}\Bigg] d_2^2 \\ & = \hbar^2 \mathcal{P}_0^2 \mathcal{C}_1^3 (s_2 \\ & + t_2) \end{aligned} \tag{2.24}$$

which implies

$$d_2^2 = \frac{\hbar^2 \mathcal{P}_0^2 \mathcal{C}_1^3 (s_2 + t_2)}{4\left[\begin{array}{c} \hbar \mathcal{P}_0 \mathcal{C}_1^2 (2+\varepsilon+3\alpha-\delta)\mathcal{Q}^{\mu}_{\varsigma,3} \\ +\hbar \mathcal{P}_0 \mathcal{C}_1^2 \left(\frac{1}{2}(\varepsilon^2+\varepsilon-2) - \delta(2\alpha-\delta) + 2\alpha(\alpha-1)\right)\mathcal{Q}^{\mu,2}_{\varsigma,2} \\ -\left((\mathcal{C}_2-\mathcal{C}_1)(1+\varepsilon+2\alpha-\delta)^2\right)\mathcal{Q}^{\mu,2}_{\varsigma,2} \end{array}\right]}. \tag{2.25}$$

Applying Lemma (1.1) in (2.25), we obtain

$$|d_2| \leq \frac{\hbar|\mathcal{P}_0|\sqrt{\mathcal{C}_1}}{\sqrt{\begin{matrix}\hbar\mathcal{P}_0\mathcal{C}_1^2(2+\varepsilon+3\alpha-\delta)\mathcal{Q}_{\varsigma,3}^{\mu} \\ +\left[\hbar\mathcal{P}_0\mathcal{C}_1^2\left(\frac{1}{2}(\varepsilon^2+\varepsilon-2)-\delta(2\alpha-\delta)+2\alpha(\alpha-1)\right)\right]\mathcal{Q}_{\varsigma,2}^{\mu,2} \\ -\left((\mathcal{C}_2-\mathcal{C}_1)(1+\varepsilon+2\alpha-\delta)^2\right)\end{matrix}}}$$

To derive the bound for $|d_3|$, we subtract (2.20) from (2.18), which gives

$$d_3 = \frac{\hbar\mathcal{C}_1\left(2\mathcal{P}_1\mathcal{s}_1+\mathcal{P}_0(\mathcal{s}_2-\mathcal{t}_2)\right)}{4(2+\varepsilon+3\alpha-\delta)\mathcal{Q}_{\varsigma,3}^{\mu}} + \frac{\hbar^2\mathcal{P}_0^2\mathcal{C}_1^2\left(\mathcal{s}_1^2+\mathcal{t}_1^2\right)}{8(1+\varepsilon+2\alpha-\delta)^2\mathcal{Q}_{\varsigma,2}^{\mu,2}}. \qquad (2.26)$$

Using Lemma (1.1) together wih (2.26), we obtain

$$|d_3| \leq \frac{\hbar\mathcal{C}_1(|\mathcal{P}_0|+|\mathcal{P}_1|)}{(2+\varepsilon+3\alpha-\delta)\mathcal{Q}_{\varsigma,3}^{\mu}} + \frac{\hbar^2\mathcal{P}_0^2\mathcal{C}_1^2}{(1+\varepsilon+2\alpha-\delta)^2\mathcal{Q}_{\varsigma,2}^{\mu,2}}$$

The proof now complete . ■

Theorem (2.1) yields the following corollary when $\delta = 0$.

Corollary(2.1): Suppose that $\mathcal{T} \in \mathcal{A}$ is member of the class $\mathcal{N}_{q,\Sigma}^{\varepsilon,\alpha}(\hbar, 0, \vartheta)$, then

$$|d_2| \leq \frac{\hbar|\mathcal{P}_0|\mathcal{C}_1\sqrt{\mathcal{C}_1}}{\sqrt{\hbar\mathcal{P}_0\mathcal{C}_1^2(2+\varepsilon+3\alpha)\mathcal{Q}_{\varsigma,3}^{\mu} + \left[\hbar\mathcal{P}_0\mathcal{C}_1^2\left(\frac{1}{2}(\varepsilon^2+\varepsilon-2)+2\alpha(\alpha-1)\right)\right]\mathcal{Q}_{\varsigma,2}^{\mu,2} - \left((\mathcal{C}_2-\mathcal{C}_1)(1+\varepsilon+2\alpha)^2\right)}},$$

and

$$|d_3| \leq \frac{\hbar\mathcal{C}_1(|\mathcal{P}_0|+|\mathcal{P}_1|)}{(2+\varepsilon+3\alpha)\mathcal{Q}_{\varsigma,3}^{\mu}} + \frac{\hbar^2\mathcal{P}_0^2\mathcal{C}_1^2}{(1+\varepsilon+2\alpha)^2\mathcal{Q}_{\varsigma,2}^{\mu,2}}.$$

Similarly, Theorem (2.1) gives he next corollary for $\hbar = 1$.

Corollary(2.2): Let $\mathcal{T} \in \mathcal{A}$ be in the class $\mathcal{N}_{q,\Sigma}^{\varepsilon,\alpha}(1, \delta, \vartheta)$, then

$$|d_2| \leq \frac{|\mathcal{P}_0|\mathcal{C}_1\sqrt{\mathcal{C}_1}}{\sqrt{\mathcal{P}_0\mathcal{C}_1^2(2+\varepsilon+3\alpha-\delta)\mathcal{Q}_{\varsigma,3}^{\mu} + \left[\mathcal{P}_0\mathcal{C}_1^2\left(\frac{1}{2}(\varepsilon^2+\varepsilon-2)-\delta(2\alpha-\delta)+2\alpha(\alpha-1)\right) - \left((\mathcal{C}_2-\mathcal{C}_1)(1+\varepsilon+2\alpha-\delta)^2\right)\right]\mathcal{Q}_{\varsigma,2}^{\mu,2}}},$$

and

$$|d_3| \leq \frac{\mathcal{C}_1(|\mathcal{P}_0|+|\mathcal{P}_1|)}{(2+\varepsilon+3\alpha-\delta)\mathcal{Q}_{\varsigma,3}^{\mu}} + \frac{\mathcal{P}_0^2\mathcal{C}_1^2}{(1+\varepsilon+2\alpha-\delta)^2\mathcal{Q}_{\varsigma,2}^{\mu,2}}.$$

Theorem (2.2): If $\mathcal{T}(\xi) \in \mathcal{N}_{q,\Sigma}^{\varepsilon,\alpha}(\hbar,\delta,\vartheta)$, then

$$\left|d_3-\mu d_2^2\right| \leq \frac{\hbar|\mathcal{C}_1|}{2(2+\varepsilon+3\alpha-\delta)\mathcal{Q}_{\varsigma,3}^{\mu}}\left[|\mathcal{P}_1| + |\mathcal{P}_0|\max\left\{1, \left|\frac{\hbar\mathcal{P}_0\mathcal{C}_1(2+\varepsilon+3\alpha-\delta)(1-\mu)}{2(1+\varepsilon+2\alpha-\delta)^2\mathcal{Q}_{\varsigma,2}^{\mu,2}}\mathcal{Q}_{\varsigma,3}^{\mu}\right|\right\}\right],$$

where $\varepsilon \geq 0, 0 \leq \hbar \leq 1, \alpha \geq 1, \delta \in \mathbb{C}$.

Proof: From equation (2.26), we have

$$d_3 = \frac{\hbar\mathcal{C}_1\left(2\mathcal{P}_1\mathcal{s}_1+\mathcal{P}_0(\mathcal{s}_2-\mathcal{t}_2)\right)}{4(2+\varepsilon+3\alpha-\delta)\mathcal{Q}_{\varsigma,3}^{\mu}} + d_2^2.$$

Therefore,

$$d_3-\mu d_2^2 = \frac{\hbar\mathcal{C}_1\left(2\mathcal{P}_1\mathcal{s}_1+\mathcal{P}_0(\mathcal{s}_2-\mathcal{t}_2)\right)}{4(2+\varepsilon+3\alpha-\delta)\mathcal{Q}_{\varsigma,3}^{\mu}} + (1-\mu)d_2^2 \qquad (2.27)$$

From equation (2.26), we have

$$d_3 - \mu d_2^2 = \frac{\hbar \mathcal{C}_1}{4(2+\varepsilon+3\alpha-\delta)\mathcal{Q}_{\varsigma,3}^{\mu}}[2\mathcal{P}_1 s_1 + \mathcal{P}_0(s_2 - t_2)] + \frac{\hbar^2 \mathcal{P}_0^2 \mathcal{C}_1^2 (s_1^2 + t_1^2)}{8(1+\varepsilon+2\alpha-\delta)^2 \mathcal{Q}_{\varsigma,2}^{\mu,2}} \quad (2.28)$$

Now put

$$\eta = \frac{\mathcal{P}_0 \mathcal{C}_1 (2+\varepsilon+3\alpha-\delta)(1-\mu)\mathcal{Q}_{\varsigma,3}^{\mu}}{2(1+\varepsilon+2\alpha-\delta)^2 \mathcal{Q}_{\varsigma,2}^{\mu,2}}$$

Thus,

$$d_3 - \mu d_2^2 = \frac{\hbar \mathcal{C}_1}{4(2+\varepsilon+3\alpha-\delta)\mathcal{Q}_{\varsigma,3}^{\mu}}\left[2\mathcal{P}_1 s_1 + \mathcal{P}_0\left(s_2 + \eta s_1^2\right) - \mathcal{P}_0\left(t_2 + \eta t_1^2\right)\right]$$

Taking absolute values,and applying Lemma (1.1), we get

$$\left|d_3 - \mu d_2^2\right| \le \frac{\hbar |\mathcal{C}_1|}{4(2+\varepsilon+3\alpha-\delta)\mathcal{Q}_{\varsigma,3}^{\mu}}\left[2|\mathcal{P}_1| + |\mathcal{P}_0|\left|s_2 + \eta s_1^2\right| + |\mathcal{P}_0|\left|t_2 + \eta t_1^2\right|\right] \quad (2.30)$$

Now applying Lemma (1.2), we get

$$\left|d_3 - \mu d_2^2\right| \le \frac{\hbar |\mathcal{C}_1|}{2(2+\varepsilon+3\alpha-\delta)\mathcal{Q}_{\varsigma,3}^{\mu}}[|\mathcal{P}_1| + |\mathcal{P}_0| \max\{1, |\eta|\}].$$

Hence, Theorem (2.2) is proved.

Taking $\delta = 0$ in Theorem (2.2), gives the next result.

Corollary(2.3): If $\mathcal{T} \in \mathcal{A}$ belongs to the class $\mathcal{N}_{q,\Sigma}^{\varepsilon,\alpha}(\hbar, 0, \vartheta)$, then

$$\left|d_3 - \mu d_2^2\right| \le \frac{\hbar |\mathcal{C}_1|}{4(2+\varepsilon+3\alpha)\mathcal{Q}_{\varsigma,3}^{\mu}}\left[2|\mathcal{P}_1| + 2|\mathcal{P}_0| \max\left\{1, \left|\frac{\hbar \mathcal{P}_0 \mathcal{C}_1 (2+\varepsilon+3\alpha)(1-\mu)}{2(1+\varepsilon+2\alpha)^2 \mathcal{Q}_{\varsigma,2}^{\mu,2}} \mathcal{Q}_{\varsigma,3}^{\mu}\right|\right\}\right].$$

3- The Subclass $\mathcal{M}_{q,\Sigma}^{\hbar}(\tau,\lambda,\vartheta)$

Definition (3.1)**:** Let $\mathcal{T} \in \mathcal{A}$ be represened by (1.1). The function $\mathcal{T}$ is in the class $\mathcal{M}_{q,\Sigma}^{\hbar}(\tau,\lambda,\vartheta)$, provied the following quasi-subordination conditions hold:

$$\frac{1}{\tau}\left[\left(\frac{\xi\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)'}{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)}\right)^{\hbar}+\frac{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)}{(1-\lambda)\xi+\lambda\xi\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)'}-1\right]\prec_q(\vartheta(\xi)-1), \tag{3.1}$$

and

$$\frac{1}{\tau}\left[\left(\frac{\zeta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)'}{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)}\right)^{\hbar}+\frac{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)}{(1-\lambda)\zeta+\lambda\zeta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)'}-1\right]\prec_q(\vartheta(\zeta)-1). \tag{3.2}$$

Where $0 \le \lambda \le 1, 0 \le \hbar \le 1, \tau \in \mathbb{C}\backslash\{0\}$, $\mathcal{R} = \mathcal{T}^{-1}$, ϑ is given by (1.6), and $\xi, \zeta \in U$.

Taking $\hbar = 0$ in Definition (3.1), yields the following special case.

Remark(3.1)**:** A function $\mathcal{T} \in \mathcal{A}$ represented by (1.1) in the class $\mathcal{M}_{q,\Sigma}(\tau,\lambda,\vartheta)$, whenever the following quasi-subordination conditions hold:

$$\frac{1}{\tau}\left[\frac{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)}{(1-\lambda)\xi+\lambda\xi\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)'}\right]\prec_q(\vartheta(\xi)-1).$$

and

$$\frac{1}{\tau}\left[\frac{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)}{(1-\lambda)\zeta+\lambda\zeta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)'}\right]\prec_q(\vartheta(\zeta)-1).$$

Theorem $(\mathbf{3.1})$: For aafunction $\mathcal{T}$ represened by (1.1) and belonging to the class $\mathcal{M}_{q,\Sigma}^{\hbar}(\tau,\lambda,\vartheta)$, the following esimates hold:

$$|d_2|$$

$$\leq\frac{\tau|\mathcal{P}_0|\mathcal{C}_1\sqrt{\mathcal{C}_1}}{\sqrt{\tau\mathcal{P}_0\mathcal{C}_1^2(1-3\lambda+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}+\left(\tau\mathcal{P}_0\mathcal{C}_1^2\left(\frac{1}{2}(\hbar^2-3\hbar)-2\lambda(1-2\lambda)\right)-(\mathcal{C}_2-\mathcal{C}_1)(1-2\lambda+\hbar)^2\right)\mathcal{Q}_{\varsigma,2}^{\mu,2}}},\quad(3.3)$$

and

$$|d_3|\leq\frac{|\tau|\mathcal{C}_1(|\mathcal{P}_0|+|\mathcal{P}_1|)}{(1-3\lambda+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}}+\frac{\tau^2\mathcal{P}_0^2\mathcal{C}_1^2}{(1-2\lambda+\hbar)^2\mathcal{Q}_{\varsigma,2}^{\mu,2}},\qquad(3.4)$$

where

$\mathcal{Q}_{\varsigma,2}^{\mu}=\left(\frac{\varsigma+1}{\varsigma+2}\right)^{\varrho}(1+(2+\rho-\mu-1)t)^{\eta}$ and

$$\mathcal{Q}_{\varsigma,3}^{\mu}=\left(\frac{\varsigma+1}{\varsigma+3}\right)^{\varrho}(1+(3+\rho-\mu-1)t)^{\eta}$$

Proof: Let $\mathcal{T}\in\mathcal{M}_{q,\Sigma}^{\hbar}(\tau,\lambda,\vartheta)$. Then there are analytic functions $\upsilon,\nu:U\to U$ satisfying $\upsilon(0)=\nu(0)=0$, $|\upsilon(\xi)|<1$ and $|\nu(\zeta)|<1$ then:

$$\frac{1}{\tau}\left[\left(\frac{\xi\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)'}{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)}\right)^{\hbar}+\frac{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)}{(1-\lambda)\xi+\lambda\xi\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)'}-1\right]$$

$$=\Psi(\xi)\big(\vartheta(\upsilon(\xi)-1)\big)\qquad(3.5)$$

and

$$\frac{1}{\tau}\left[\left(\frac{\zeta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)'}{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)}\right)^{\hbar}+\frac{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)}{(1-\lambda)\zeta+\lambda\zeta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)'}-1\right]$$
$$=\Psi(\zeta)\big(\vartheta(\nu(\zeta)-1)\big) \tag{3.6}$$

where $\mathcal{R}=\mathcal{T}^{-1}$ and $\xi,\zeta\in U$.

Let $\wp(\xi)$ and $q(\zeta)$ be defined by (2.7) and (2.8) respectively.

$$\frac{1}{\tau}\left[\left(\frac{\xi\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)'}{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)}\right)^{\hbar}+\frac{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)}{(1-\lambda)\xi+\lambda\xi\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)'}-1\right]$$
$$=\Psi(\xi)\left(\vartheta\left(\frac{\wp(\xi)-1}{\wp(\xi)+1}\right)-1\right), \tag{3.7}$$

and

$$\frac{1}{\tau}\left[\left(\frac{\zeta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)'}{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)}\right)^{\hbar}+\frac{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)}{(1-\lambda)\zeta+\lambda\zeta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)'}-1\right]$$
$$=\Psi(\zeta)\left(\vartheta\left(\frac{q(\zeta)-1}{q(\zeta)+1}\right)-1\right). \tag{3.8}$$

Using the series expansions for $\mathcal{T}(\xi)$ and $\mathcal{R}(\zeta)$ given by (1.1) and (1.2) respectively, we obtain

$$\frac{1}{\tau}\left[\left(\frac{\xi\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)'}{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)}\right)^{\hbar}+\frac{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)}{(1-\lambda)\xi+\lambda\xi\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{T}(\xi)\right)'}-1\right]$$

$$=\frac{1}{\tau}\Bigg[1+(1-2\lambda+\hbar)\mathcal{Q}_{\varsigma,2}^{\mu}d_2\xi$$

$$+\Bigg((1-3\lambda+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}d_3$$

$$+\left(\frac{1}{2}(\hbar^2-3\hbar)-2\lambda(1-2\lambda)\right)\mathcal{Q}_{\varsigma,2}^{\mu,2}d_2^2\Bigg)\xi^2$$

$$+\cdots\Bigg]. \qquad (3.9)$$

and

$$\frac{1}{\tau}\left[\left(\frac{\zeta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)'}{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)}\right)^{\hbar}+\frac{\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)}{(1-\lambda)\zeta+\lambda\zeta\left(\mathcal{H}_{\mu,\varsigma,t}^{\varrho,\eta,\rho}\mathcal{R}(\zeta)\right)'}-1\right]$$

$$=\frac{1}{\tau}\Bigg[1-(1-2\lambda+\hbar)\mathcal{Q}_{\varsigma,2}^{\mu}d_2\zeta$$

$$+\Bigg((1-3\lambda+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}\left(2d_2^2-d_3\right)+\left(\frac{1}{2}(\hbar^2-3\hbar)-2\lambda(1-2\lambda)\right)\mathcal{Q}_{\varsigma,2}^{\mu,2}d_2^2\Bigg)\zeta^2$$

$$+\cdots\Bigg]. \qquad (3.10)$$

A comparison of the coefficients of ξ and ξ^2 in (3.9) and (2.15), yields

$$\frac{1}{\tau}(1-2\lambda+\hbar)\mathcal{Q}_{\varsigma,2}^{\mu}d_2=\frac{1}{2}\mathcal{P}_0\mathcal{C}_1\mathcal{s}_1 \qquad (3.11)$$

and

$$\frac{1}{\tau}\left[(1-3\lambda+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}d_3+\left(\frac{1}{2}(\hbar^2-3\hbar)-2\lambda(1-2\lambda)\right)\mathcal{Q}_{\varsigma,2}^{\mu,2}d_2^2\right]$$
$$=\frac{1}{2}\mathcal{P}_1\mathcal{C}_1\mathcal{s}_1+\frac{1}{2}\mathcal{P}_0\mathcal{C}_1\left(\mathcal{s}_2-\frac{\mathcal{s}_1^2}{2}\right)+\frac{1}{4}\mathcal{P}_0\mathcal{C}_2\mathcal{s}_1^2 \tag{3.12}$$

In the same way, comparing the coefficients of ζ and ζ^2 in (3.10) and (2.16), gives

$$-\frac{1}{\tau}(1-2\lambda+\hbar)\mathcal{Q}_{\varsigma,2}^{\mu}d_2=\frac{1}{2}\mathcal{P}_0\mathcal{C}_1\mathcal{t}_1 \tag{3.13}$$

$$\frac{1}{\tau}\left[\left((1-3\lambda+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}(2d_2^2-d_3)+\left(\frac{1}{2}(\hbar^2-3\hbar)-2\lambda(1-2\lambda)\right)\mathcal{Q}_{\varsigma,2}^{\mu,2}d_2^2\right)\right]$$
$$=\frac{1}{2}\mathcal{P}_1\mathcal{C}_1\mathcal{t}_1+\frac{1}{2}\mathcal{P}_0\mathcal{C}_1\left(\mathcal{t}_2-\frac{\mathcal{t}_1^2}{2}\right)+\frac{1}{4}\mathcal{P}_0\mathcal{C}_2\mathcal{t}_1^2 \tag{3.14}$$

Equations (3.11) and (3.13), give

$$d_2=\frac{\tau\mathcal{P}_0\mathcal{C}_1\mathcal{s}_1}{2(1-2\lambda+\hbar)\mathcal{Q}_{\varsigma,2}^{\mu}}=-\frac{\tau\mathcal{P}_0\mathcal{C}_1\mathcal{t}_1}{2(1-2\lambda+\hbar)\mathcal{Q}_{\varsigma,2}^{\mu}} \tag{3.15}$$

It follows that

$$\mathcal{s}_1=-\mathcal{t}_1 \tag{3.16}$$

and

$$8(1-2\lambda+\hbar)^2\mathcal{Q}_{\varsigma,2}^{\mu,2}d_2^2=\tau^2\mathcal{P}_0^2\mathcal{C}_1^2\left(\mathcal{s}_1^2+\mathcal{t}_1^2\right) \tag{3.17}$$

Applying Lemma (1.1) in (3.15) we obtain

$$|d_2|\leq\frac{|\tau\mathcal{P}_0|\mathcal{C}_2}{(1-2\lambda+\hbar)\mathcal{Q}_{\varsigma,2}^{\mu}}.$$

Adding (3.12) and (3.14), and using (3.16) and (3.17), we obtain

$$4\left[\tau\mathcal{P}_0\mathcal{C}_1^2(1-3\lambda+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}+\left(\frac{1}{2}(\hbar^2-3\hbar)-2\lambda(1-2\lambda)\right)\mathcal{Q}_{\varsigma,2}^{\mu,2}\right.$$
$$\left.-(\mathcal{C}_2-\mathcal{C}_1)(1-2\lambda+\hbar)^2\mathcal{Q}_{\varsigma,2}^{\mu,2}\right]d_2^2$$
$$=\tau^2\mathcal{P}_0^2\mathcal{C}_1^3(\mathcal{s}_2+\mathcal{t}_2) \qquad (3.18)$$

which implies

$$d_2^2 = \frac{\tau^2\mathcal{P}_0^2\mathcal{C}_1^3(\mathcal{s}_2+\mathcal{t}_2)}{4\left[\begin{array}{c}\tau\mathcal{P}_0\mathcal{C}_1^2(1-3\lambda+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}+\\ \left(\tau\mathcal{P}_0\mathcal{C}_1^2\left(\frac{1}{2}(\hbar^2-3\hbar)-2\lambda(1-2\lambda)\right)-(\mathcal{C}_2-\mathcal{C}_1)(1-2\lambda+\hbar)^2\right)\mathcal{Q}_{\varsigma,2}^{\mu,2}\end{array}\right]}. \qquad (3.19)$$

Applying Lemma (1.1) in (3.19), we obtain

$$|d_2| \leq \frac{|\tau\mathcal{P}_0|\mathcal{C}_1\sqrt{\mathcal{C}_1}}{\sqrt{\begin{array}{c}\tau\mathcal{P}_0\mathcal{C}_1^2(1-3\lambda+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}+\\ \left(\tau\mathcal{P}_0\mathcal{C}_1^2\left(\frac{1}{2}(\hbar^2-3\hbar)-2\lambda(1-2\lambda)\right)-(\mathcal{C}_2-\mathcal{C}_1)(1-2\lambda+\hbar)^2\right)\mathcal{Q}_{\varsigma,2}^{\mu,2}\end{array}}}$$

Furthermore, to obtain the bound for $|d_3|$, subtracting (3.12) and (3.14), we obtain

$$d_3 = \frac{\tau\mathcal{C}_1\left(2\mathcal{P}_1\mathcal{s}_1+\mathcal{P}_0(\mathcal{s}_2-\mathcal{t}_2)\right)}{4(1-3\lambda+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}} + \frac{\tau^2\mathcal{P}_0^2\mathcal{C}_1^2\left(\mathcal{s}_1^2+\mathcal{t}_1^2\right)}{8(1-2\lambda+\hbar)^2\mathcal{Q}_{\varsigma,2}^{\mu,2}} \qquad (3.20)$$

Applying Lemma (1.1) in (3.20), we get

$$|d_3| \leq \frac{|\tau|\mathcal{C}_1(|\mathcal{P}_0| + |\mathcal{P}_1|)}{(1-3\lambda+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}} + \frac{\tau^2\mathcal{P}_0^2\mathcal{C}_1^2}{(1-2\lambda+\hbar)^2\mathcal{Q}_{\varsigma,2}^{\mu,2}}$$

This complete the proof. ■

Theorem (3.1) reduces to the next corollary when $\lambda = 0$.

Corollary (3.1): If $\mathcal{T} \in \mathcal{A}$ belongs to the class $\mathcal{M}_{q,\Sigma}^{\hbar}(\tau, \vartheta)$, then

$$|d_2|$$

$$\leq \frac{|\tau\mathcal{P}_0|\mathcal{C}_1\sqrt{\mathcal{C}_1}}{\sqrt{\tau\mathcal{P}_0\mathcal{C}_1^2(1+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu} + \left(\tau\mathcal{P}_0\mathcal{C}_1^2\left(\frac{1}{2}(\hbar^2-3\hbar)\right) - (\mathcal{C}_2-\mathcal{C}_1)(1+\hbar)^2\right)\mathcal{Q}_{\varsigma,2}^{\mu,2}}}.$$

and

$$|d_3| \leq \frac{|\tau|\mathcal{C}_1(|\mathcal{P}_0| + |\mathcal{P}_1|)}{(1+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}} + \frac{\tau^2\mathcal{P}_0^2\mathcal{C}_1^2}{(1+\hbar)^2\mathcal{Q}_{\varsigma,2}^{\mu,2}}$$

Similarly, takin $\hbar = 0$ and $\tau = 1$ in Theorem (3.1) gives the next corollary.

Corollary (3.2): Suppose that $\mathcal{T} \in \mathcal{A}$ belonge to the class $\mathcal{M}_{\mathrm{q},\Sigma}(\tau, \lambda, \vartheta)$, then

$$|d_2| \leq \frac{|\mathcal{P}_0|\mathcal{C}_1\sqrt{\mathcal{C}_1}}{\sqrt{\tau\mathcal{P}_0\mathcal{C}_1^2(1-3\lambda)\mathcal{Q}_{\varsigma,3}^{\mu} - (\mathcal{C}_2-\mathcal{C}_1)(1-2\lambda)^2\mathcal{Q}_{\varsigma,2}^{\mu,2}}}.$$

and

$$|d_3| \leq \frac{\mathcal{C}_1(|\mathcal{P}_0| + |\mathcal{P}_1|)}{(1-3\lambda)\mathcal{Q}_{\varsigma,3}^{\mu}} + \frac{\mathcal{P}_0^2\mathcal{C}_1^2}{(1-2\lambda)^2\mathcal{Q}_{\varsigma,2}^{\mu,2}}$$

Theorem (3.2): If $\mathcal{T}(\xi) \in \mathcal{M}_{q,\Sigma}^{\hbar}(\tau, \lambda, \vartheta)$, then

$$\left|d_3 - \mu d_2^2\right| \leq \frac{\tau|\mathcal{C}_1|}{2(1-3\lambda+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}}\left[|\mathcal{P}_1| + |\mathcal{P}_0| \max\left\{1, \left|\frac{\tau\mathcal{P}_0\mathcal{C}_1(1-3\lambda+2\hbar)(1-\mu)\mathcal{Q}_{\varsigma,3}^{\mu}}{2(1-2\lambda+\hbar)^2\mathcal{Q}_{\varsigma,2}^{\mu,2}}\right|\right\}\right],$$

where $0 \leq \lambda \leq 1, 0 \leq \hbar \leq 1, \tau \in \mathbb{C}\backslash\{0\}$.

Proof: From equation (3.20), we have

$$d_3 = \frac{\tau\mathcal{C}_1\left(2\mathcal{P}_1\mathcal{s}_1 + \mathcal{P}_0(\mathcal{s}_2 - \mathcal{t}_2)\right)}{4(1-3\lambda+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}} + d_2^2$$

Therefore,

$$d_3 - \mu d_2^2 = \frac{\tau\mathcal{C}_1\left(2\mathcal{P}_1\mathcal{s}_1 + \mathcal{P}_0(\mathcal{s}_2 - \mathcal{t}_2)\right)}{4(1-3\lambda+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}} + (1-\mu)d_2^2 \tag{3.21}$$

From equation (3.17), we have

$$d_3 - \mu d_2^2 = \frac{\tau\mathcal{C}_1}{4(1-3\lambda+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}}[2\mathcal{P}_1\mathcal{s}_1 + \mathcal{P}_0(\mathcal{s}_2 - \mathcal{t}_2)] + \frac{\tau^2\mathcal{P}_0^2\mathcal{C}_1^2\left(\mathcal{s}_1^2 + \mathcal{t}_1^2\right)}{8(1-2\lambda+\hbar)^2\mathcal{Q}_{\varsigma,2}^{\mu,2}}. \tag{3.22}$$

Now put

$$\eta = \frac{\tau\mathcal{P}_0\mathcal{C}_1(1-3\lambda+2\hbar)(1-\mu)\mathcal{Q}_{\varsigma,3}^{\mu}}{2(1-2\lambda+\hbar)^2\mathcal{Q}_{\varsigma,2}^{\mu,2}}$$

Thus,

$$d_3 - \mu d_2^2 = \frac{\tau \mathcal{C}_1}{4(1-3\lambda+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}}\left[2\mathcal{P}_1 s_1 + \mathcal{P}_0\left(s_2 + \eta s_1^2\right)\right.$$
$$\left. - \mathcal{P}_0\left(t_2 + \eta t_1^2\right)\right]. \qquad (3.23)$$

Taking absolute values,and applying Lemma (1.1), we obtain

$$\left|d_3 - \mu d_2^2\right| \leq \frac{\tau|\mathcal{C}_1|}{4(1-3\lambda+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}}\left[2|\mathcal{P}_1| + |\mathcal{P}_0|\left|s_2 + \eta s_1^2\right| + |\mathcal{P}_0|\left|t_2 + \eta t_1^2\right|\right]$$

Now applying Lemma (1.2), we get

$$\left|d_3 - \mu d_2^2\right| \leq \frac{\tau|\mathcal{C}_1|}{2(1-3\lambda+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}}\left[|\mathcal{P}_1| + |\mathcal{P}_0|\max\{1,|\eta|\}\right]$$

Hence, Theorem (3.2) is proved.

For $\lambda = 0$ in Theorem (3.2), the next result follows.

Corollary(3.3): If $\mathcal{T} \in \mathcal{A}$ is a member of the class $\mathcal{M}_{q,\Sigma}^{\hbar}(\tau,\vartheta)$, then

$$\left|d_3 - \mu d_2^2\right| \leq \frac{\tau|\mathcal{C}_1|}{2(1+2\hbar)\mathcal{Q}_{\varsigma,3}^{\mu}}\left[|\mathcal{P}_1|\right.$$
$$\left. + |\mathcal{P}_0|\max\left\{1, \left|\frac{\tau\mathcal{P}_0\mathcal{C}_1(1+2\hbar)(1-\mu)\mathcal{Q}_{\varsigma,3}^{\mu}}{2(1+\hbar)^2\mathcal{Q}_{\varsigma,2}^{\mu,2}}\right|\right\}\right]$$

Conclusions

In conclusion, a new convolution operator based on the convolution of the Komatu integral operator and the Opoola differential operator was considered. Using this operator and quasi-subordination, new subclasses of bi-univalent functions were

introduced. For these subclasses, estimates of the initial coefficients $|d_2|$ and $|d_3|$, as well as the corresponding Fekete–Szegö inequalities, were obtained.

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